

# Simple proofs of Jensen's, Chu's, Mohanty-Handa's, and Graham-Knuth-Patashnik's identities

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**Abstract.** Motivated by the recent work of Chu [Electron. J. Combin. 17 (2010), #N24], we give simple proofs of Jensen's identity

$$\sum_{k=0}^n \binom{x+kz}{k} \binom{y-kz}{n-k} = \sum_{k=0}^n \binom{x+y-k}{n-k} z^k,$$

and Chu's and Mohanty-Handa's generalizations of Jensen's identity. We also give a quite simple proof of an equivalent form of Graham-Knuth-Patashnik's identity

$$\sum_{k \geq 0} \binom{m+r}{m-n-k} \binom{n+k}{n} x^{m-n-k} y^k = \sum_{k \geq 0} \binom{-r}{m-n-k} \binom{n+k}{n} (-x)^{m-n-k} (x+y)^k,$$

which was rediscovered, respectively, by Sun in 2003 and Munarini in 2005. Finally we give a multinomial coefficient generalization of this identity and raise two open problems.

*Keywords:* Jensen's identity, Chu's identity, Mohanty-Handa's identity, Graham-Knuth-Patashnik's, Chu-Vandermonde, multinomial coefficient

## 1 Introduction

Abel's identity (see, for example, [8, §3.1])

$$\sum_{k=0}^n \binom{n}{k} x(x+kz)^{k-1} (y-kz)^{n-k} = (x+y)^n \quad (1.1)$$

and Rothe's identity (or called Hagen-Rothe's identity, see, for example, [9, §5.4])

$$\sum_{k=0}^n \frac{x}{x-kz} \binom{x-kz}{k} \binom{y+kz}{n-k} = \binom{x+y}{n}, \quad (1.2)$$

are famous in the literature and play an important role in enumerative combinatorics. Recently, Chu [6] gave elementary proofs of Abel's identity and Rothe's identity by using the binomial theorem and the Chu-Vandermonde convolution formula respectively.

Motivated by Chu's work, we shall study Jensen's identity [17], which is closely related to Rothe's identity, and can be stated as follows:

$$\sum_{k=0}^n \binom{x+kz}{k} \binom{y-kz}{n-k} = \sum_{k=0}^n \binom{x+y-k}{n-k} z^k. \quad (1.3)$$

Jensen's identity (1.3) has ever attracted much attention by different authors. Gould [11] obtained the following Abel-type analogue:

$$\sum_{k=0}^n \frac{(x+kz)^k}{k!} \frac{(y-kz)^{n-k}}{(n-k)!} = \sum_{k=0}^n \frac{(x+y)^k}{k!} z^{n-k}. \quad (1.4)$$

Carlitz [1] gave two interesting theorem related to (1.3) and (1.4) by mathematical induction. With the help of generating functions, Gould [12] derived the following variation of Jensen's identity (1.3):

$$\sum_{k=0}^n \binom{x+kz}{k} \binom{y-kz}{n-k} = \sum_{k=0}^n k \binom{x+y-k}{n-k} \frac{x+y-(n-k)z-k}{x+y-k} z^k. \quad (1.5)$$

E. G.-Rodeja F. [10] deduced Gould's identity (1.4) from (1.3) by establishing an identity which includes both. Cohen and Sun [7] also gave an expression which unifies (1.3) and (1.4). Chu [4] generalized Jensen's identity (1.3) to a multi-sum form:

$$\sum_{k_1+\dots+k_s=n} \prod_{i=1}^s \binom{x_i+k_i z}{k_i} = \sum_{k=0}^n \binom{k+s-2}{k} \binom{x_1+\dots+x_s+nz-k}{n-k} z^k. \quad (1.6)$$

Moreover, the identities (1.3) and (1.6) were respectively generalized by Mohanty and Handa [20] and Chu [5] to the case of multinomial coefficients (to be stated in Section 4).

The first purpose of this paper is to give simple proofs of Jensen's identity, Chu's identity (1.6), Mohanty-Handa's identity, and Chu's generalization of Mohanty-Handa's identity. We shall use the Chu-Vandermonde convolution formula

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}$$

and a well-known identity

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k^r = \begin{cases} 0, & \text{if } 0 \leq r \leq n-1, \\ n!, & \text{if } r = n. \end{cases} \quad (1.7)$$

Eq. (1.7) may be easily deduced from the Stirling numbers of the second kind [28, p. 34, (24a)]. The first case of (1.7) was already utilized by the author [13] to give a simple proof of Dixon's identity and by Chu [6] in his proofs of Abel's and Rothe's identities.

It is interesting that our proof of Chu's identity (1.6) will also leads to a very short proof of Graham-Knuth-Patashnik's identity, which was rediscovered several times in the past few years. The second purpose of this paper is to give a multinomial coefficient generalization of Graham-Knuth-Patashnik's identity and raise two open problems.

## 2 Proof of Jensen's identity

By the Chu-Vandermonde convolution formula, we have

$$\sum_{k=0}^n \binom{x+kz}{k} \binom{y-kz}{n-k} = \sum_{k=0}^n \binom{x+kz}{k} \sum_{i=k}^n \binom{x+y+1}{n-i} \binom{-x-kz-1}{i-k} \quad (2.1)$$

Interchanging the summation order in (2.1) and noticing that

$$\binom{x+kz}{k} \binom{-x-kz-1}{i-k} = (-1)^{i-k} \binom{i}{k} \binom{x+kz+i-k}{i},$$

we have

$$\begin{aligned} \sum_{k=0}^n \binom{x+kz}{k} \binom{y-kz}{n-k} &= \sum_{i=0}^n \binom{x+y+1}{n-i} \sum_{k=0}^i (-1)^{i-k} \binom{i}{k} \binom{x+kz+i-k}{i} \\ &= \sum_{i=0}^n \binom{x+y+1}{n-i} (z-1)^i, \end{aligned} \quad (2.2)$$

where the second equality holds because  $\binom{x+kz+i-k}{i}$  is a polynomial in  $k$  of degree  $i$  with leading coefficient  $(z-1)^i/i!$  and we can apply (1.7) to simplify. We now substitute  $x \rightarrow -x-1$ ,  $y \rightarrow -y+n-1$  and  $z \rightarrow -z+1$  in (2.2) and observe that

$$\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}. \quad (2.3)$$

Then we obtain

$$\sum_{k=0}^n \binom{x+kz}{k} \binom{y-kz}{n-k} = \sum_{i=0}^n \binom{x+y-i}{n-i} z^i, \quad (2.4)$$

as desired.

Combining (1.3) and (2.2), we get the following identity:

$$\sum_{k=0}^n \binom{x-k}{n-k} z^k = \sum_{k=0}^n \binom{x+1}{n-k} (z-1)^k,$$

which is equivalent to the following identity in Graham et al. [9, p. 218]:

$$\sum_{k \leq m} \binom{m+r}{k} x^k y^{m-k} = \sum_{k \leq m} \binom{-r}{k} (-x)^k (x+y)^{m-k}.$$

### 3 Proofs of Chu's and Graham-Knuth-Patashnik's identities

By repeatedly using the Chu-Vandermonde convolution formula, we have

$$\begin{aligned} \binom{x_s + k_s z}{k_s} &= \binom{x_s + (n - k_1 - \dots - k_{s-1})z}{n - k_1 - \dots - k_{s-1}} \\ &= \sum_{j=k_1+\dots+k_{s-1}}^n \sum_{j_1+\dots+j_{s-1}=j} \binom{x_1 + \dots + x_s + nz + s - 1}{n-j} \\ &\quad \times \prod_{i=1}^{s-1} \binom{-x_i - k_i z - 1}{j_i - k_i}. \end{aligned} \quad (3.1)$$

It follows that

$$\begin{aligned} \sum_{k_1+\dots+k_s=n} \prod_{i=1}^s \binom{x_i + k_i z}{k_i} &= \sum_{k_1+\dots+k_{s-1}=0}^n \sum_{j=k_1+\dots+k_{s-1}}^n \sum_{j_1+\dots+j_{s-1}=j} \binom{x_1+\dots+x_s+nz+s-1}{n-j} \\ &\quad \times \prod_{i=1}^{s-1} \binom{x_i + k_i z}{k_i} \binom{-x_i - k_i z - 1}{j_i - k_i}. \end{aligned} \quad (3.2)$$

Interchanging the summation order in (3.2) and observing that

$$\binom{x_i + k_i z}{k_i} \binom{-x_i - k_i z - 1}{j_i - k_i} = (-1)^{j_i - k_i} \binom{j_i}{k_i} \binom{x_i + k_i z + j_i - k_i}{j_i}$$

and  $\binom{x_i + k_i z + j_i - k_i}{j_i}$  is a polynomial in  $k_i$  of degree  $j_i$  with leading coefficient  $(z-1)^{j_i}/j_i!$ , by (1.7) we get

$$\begin{aligned} \sum_{k_1+\dots+k_s=n} \prod_{i=1}^s \binom{x_i + k_i z}{k_i} &= \sum_{j=0}^n \binom{x_1+\dots+x_s+nz+s-1}{n-j} \sum_{j_1+\dots+j_{s-1}=j} (z-1)^j \\ &= \sum_{j=0}^n \binom{j+s-2}{j} \binom{x_1+\dots+x_s+nz+s-1}{n-j} (z-1)^j. \end{aligned} \quad (3.3)$$

Substituting  $x_i \rightarrow -x_i - 1$  ( $i = 1, \dots, s$ ) and  $z \rightarrow -z + 1$  in (3.3) and using (2.3), we immediately get Chu's identity (1.6).

Comparing (1.6) with (3.3) and replacing  $s$  by  $s + 2$ , we immediately get

$$\sum_{k=0}^n \binom{k+s}{k} \binom{x-k}{n-k} z^k = \sum_{j=0}^n \binom{k+s}{k} \binom{x+s+1}{n-k} (z-1)^k. \quad (3.4)$$

It is easy to see that the identity (3.4) is equivalent to each of the following known identities:

- Graham-Knuth-Patashnik's identity [9, p. 218]

$$\sum_{k \geq 0} \binom{m+r}{m-n-k} \binom{n+k}{n} x^{m-n-k} y^k = \sum_{k \geq 0} \binom{-r}{m-n-k} \binom{n+k}{n} (-x)^{m-n-k} (x+y)^k.$$

- Sun's identity [30]

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \binom{n+k}{a} (1+x)^{n+k-a} = \sum_{k=0}^n \binom{n}{k} \binom{m+k}{a} x^{m+k-a}. \quad (3.5)$$

- Munarini's identity [21]

$$\sum_{k=0}^n (-1)^{n-k} \binom{\beta - \alpha + n}{n-k} \binom{\beta + k}{k} (1+x)^k = \sum_{k=0}^n \binom{\alpha}{n-k} \binom{\beta + k}{k} x^k. \quad (3.6)$$

Moreover, the following special case

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} (1+x)^k = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} x^k \quad (3.7)$$

was reproved by Simons [27], Hirschhorn [16], Chapman [2], Prodinger [22], Wang and Sun [31].

## 4 Mohanty-Handa's identity and Chu's generalization

Let  $m$  be a fixed positive integer. For  $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{N}^m$  and  $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{C}^m$ , set  $|\mathbf{a}| = a_1 + \dots + a_m$ ,  $\mathbf{a}! = a_1! \cdots a_m!$ ,  $\mathbf{a} + \mathbf{b} = (a_1 + b_1, \dots, a_m + b_m)$ ,  $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \dots + a_m b_m$ , and  $\mathbf{b}^{\mathbf{a}} = b_1^{a_1} \cdots b_m^{a_m}$ . For any variable  $x$  and  $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{Z}^m$ , the *multinomial coefficient*  $\binom{x}{\mathbf{n}}$  is defined by

$$\binom{x}{\mathbf{n}} = \begin{cases} x(x-1) \cdots (x - |\mathbf{n}| + 1) / \mathbf{n}!, & \text{if } \mathbf{n} \in \mathbb{N}^m, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we let  $\mathbf{0} = (0, \dots, 0)$  and  $\mathbf{1} = (1, \dots, 1)$ .

In 1969, Mohanty and Handa [20] established the following multinomial coefficient generalization of Jensen's identity

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{x + \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{y - \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{x + y - |\mathbf{k}|}{\mathbf{n} - \mathbf{k}} \binom{|\mathbf{k}|}{\mathbf{k}} \mathbf{z}^{\mathbf{k}}. \quad (4.1)$$

Twenty years later, Mohanty-Handa's identity was generalized by Chu [5] as follows:

$$\sum_{\mathbf{k}_1 + \dots + \mathbf{k}_s = \mathbf{n}} \prod_{i=1}^s \binom{x_i + \mathbf{k}_i \cdot \mathbf{z}}{\mathbf{k}_i} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{|\mathbf{k}| + s - 2}{\mathbf{k}} \binom{x_1 + \dots + x_s + \mathbf{n} \cdot \mathbf{z} - |\mathbf{k}|}{\mathbf{n} - \mathbf{k}} \mathbf{z}^{\mathbf{k}}, \quad (4.2)$$

which is also a generalization of (1.6).

*Remark.* Note that the corresponding multinomial coefficient generalization of Rothe's identity was already obtained by Raney [23] (for a special case) and Mohanty [18]. The reader is referred to Strehl [29] for a historical note on Raney-Mohanty's identity.

In what follows, we will give an elementary proof of Chu's identity (4.2) similar to that of (1.6). First note that the Chu-Vandermonde convolution formula has the following trivial generalization

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{x}{\mathbf{k}} \binom{y}{\mathbf{n} - \mathbf{k}} = \binom{x + y}{\mathbf{n}}, \quad (4.3)$$

as mentioned by Zeng [32], while (1.7) can be easily generalized as

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} (-1)^{|\mathbf{n}| - |\mathbf{k}|} \binom{\mathbf{n}}{\mathbf{k}} \mathbf{k}^{\mathbf{r}} = \begin{cases} 0, & \text{if } r_i < n_i \text{ for some } 1 \leq i \leq m. \\ \mathbf{n}!, & \text{if } \mathbf{r} = \mathbf{n}, \end{cases} \quad (4.4)$$

where

$$\binom{\mathbf{n}}{\mathbf{k}} := \prod_{i=1}^m \binom{n_i}{k_i}.$$

**Lemma 4.1** *For  $\mathbf{n} \in \mathbb{N}^m$  and  $s \geq 1$ , there holds*

$$\sum_{\mathbf{k}_1 + \dots + \mathbf{k}_s = \mathbf{n}} \prod_{i=1}^s \binom{|\mathbf{k}_i|}{\mathbf{k}_i} = \binom{|\mathbf{n}| + s - 1}{\mathbf{n}}. \quad (4.5)$$

*Proof.* For nonnegative integers  $a_1, \dots, a_s$  such that  $a_1 + \dots + a_s = |\mathbf{n}|$ , by the Chu-Vandermonde convolution formula (4.4), the following identity holds

$$\sum_{\mathbf{k}_1 + \dots + \mathbf{k}_s = \mathbf{n}} \prod_{i=1}^s \binom{a_i}{\mathbf{k}_i} = \binom{|\mathbf{n}|}{\mathbf{n}}. \quad (4.6)$$

Moreover, in this case, for  $\mathbf{k}_1 + \dots + \mathbf{k}_s = \mathbf{n}$ , we have

$$\prod_{i=1}^s \binom{a_i}{\mathbf{k}_i} \neq 0 \quad \text{if and only if} \quad |\mathbf{k}_i| = a_i \quad (i = 1, \dots, s).$$

Thus, the identity (4.6) may be rewritten as

$$\sum_{\substack{\mathbf{k}_1 + \dots + \mathbf{k}_s = \mathbf{n} \\ |\mathbf{k}_1| = a_1, \dots, |\mathbf{k}_s| = a_s}} \prod_{i=1}^s \binom{a_i}{\mathbf{k}_i} = \binom{|\mathbf{n}|}{\mathbf{n}}.$$

It follows that

$$\begin{aligned} \sum_{\mathbf{k}_1 + \dots + \mathbf{k}_s = \mathbf{n}} \prod_{i=1}^s \binom{|\mathbf{k}_i|}{\mathbf{k}_i} &= \sum_{a_1 + \dots + a_s = |\mathbf{n}|} \sum_{\substack{\mathbf{k}_1 + \dots + \mathbf{k}_s = \mathbf{n} \\ |\mathbf{k}_1| = a_1, \dots, |\mathbf{k}_s| = a_s}} \prod_{i=1}^s \binom{a_i}{\mathbf{k}_i} \\ &= \sum_{a_1 + \dots + a_s = |\mathbf{n}|} \binom{|\mathbf{n}|}{\mathbf{n}} \\ &= \binom{|\mathbf{n}| + s - 1}{|\mathbf{n}|} \binom{|\mathbf{n}|}{\mathbf{n}}, \end{aligned}$$

as desired. ■

By repeatedly using the convolution formula (4.3), we may rewrite the left-hand side of (4.2) as

$$\begin{aligned} &\sum_{\mathbf{k}_1 + \dots + \mathbf{k}_{s-1} = \mathbf{0}}^{\mathbf{n}} \sum_{\mathbf{j} = \mathbf{k}_1 + \dots + \mathbf{k}_{s-1}}^{\mathbf{n}} \sum_{\mathbf{j}_1 + \dots + \mathbf{j}_{s-1} = \mathbf{j}} \binom{x_1 + \dots + x_s + \mathbf{n} \cdot \mathbf{z} + m - 1}{\mathbf{n} - \mathbf{j}} \\ &\quad \times \prod_{i=1}^{s-1} \binom{x_i + \mathbf{k}_i \cdot \mathbf{z}}{\mathbf{k}_i} \binom{-x_i - \mathbf{k}_i \cdot \mathbf{z} - 1}{\mathbf{j}_i - \mathbf{k}_i}. \end{aligned} \quad (4.7)$$

Interchanging the summation order in (4.7), observing that

$$\binom{x_i + \mathbf{k}_i \cdot \mathbf{z}}{\mathbf{k}_i} \binom{-x_i - \mathbf{k}_i \cdot \mathbf{z} - 1}{\mathbf{j}_i - \mathbf{k}_i} = (-1)^{|\mathbf{j}_i| - |\mathbf{k}_i|} \binom{\mathbf{j}_i}{\mathbf{k}_i} \binom{x_i + \mathbf{k}_i \cdot \mathbf{z} + |\mathbf{j}_i| - |\mathbf{k}_i|}{\mathbf{j}_i}$$

and

$$\binom{x_i + \mathbf{k}_i \cdot \mathbf{z} + |\mathbf{j}_i| - |\mathbf{k}_i|}{\mathbf{j}_i}$$

is a polynomial in  $k_{i1}, \dots, k_{im}$  with the coefficient of  $\mathbf{k}_i^{\mathbf{j}_i}$  being  $\binom{|\mathbf{j}_i|}{\mathbf{j}_i}(\mathbf{z}-\mathbf{1})^{\mathbf{j}_i}/\mathbf{j}_i!$ . Applying (4.4), we get

$$\begin{aligned} & \sum_{\mathbf{k}_1+\dots+\mathbf{k}_s=\mathbf{n}} \prod_{i=1}^s \binom{x_i + \mathbf{k}_i \cdot \mathbf{z}}{\mathbf{k}_i} \\ &= \sum_{\mathbf{j}=0}^{\mathbf{n}} \binom{x_1 + \dots + x_s + \mathbf{n} \cdot \mathbf{z} + s - 1}{\mathbf{n} - \mathbf{j}} (\mathbf{z} - \mathbf{1})^{\mathbf{j}} \sum_{\mathbf{j}_1+\dots+\mathbf{j}_{s-1}=\mathbf{j}} \prod_{i=1}^m \binom{|\mathbf{j}_i|}{\mathbf{j}_i} \\ &= \sum_{\mathbf{j}=0}^{\mathbf{n}} \binom{|\mathbf{j}| + s - 2}{\mathbf{j}} \binom{x_1 + \dots + x_s + \mathbf{n} \cdot \mathbf{z} + s - 1}{\mathbf{n} - \mathbf{j}} (\mathbf{z} - \mathbf{1})^{\mathbf{j}}, \end{aligned} \quad (4.8)$$

where the second equality follows from (4.5). Substituting  $x_i \rightarrow -x_i - 1$  ( $i = 1, \dots, s$ ) and  $\mathbf{z} \rightarrow -\mathbf{z} + \mathbf{1}$  in (4.8) and observing that  $\binom{-x}{\mathbf{k}} = (-1)^{|\mathbf{k}|} \binom{x+|\mathbf{k}|-1}{\mathbf{k}}$ , we immediately get (4.2).

Comparing (4.2) with (4.8) and replacing  $s$  by  $s + 2$ , we obtain the following result.

**Theorem 4.2** *For  $\mathbf{n} \in \mathbb{N}^m$  and  $\mathbf{z} \in \mathbb{C}^m$ , there holds*

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{|\mathbf{k}| + s}{\mathbf{k}} \binom{x - |\mathbf{k}|}{\mathbf{n} - \mathbf{k}} \mathbf{z}^{\mathbf{k}} = \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{|\mathbf{k}| + s}{\mathbf{k}} \binom{x + s + 1}{\mathbf{n} - \mathbf{k}} (\mathbf{z} - \mathbf{1})^{\mathbf{k}}. \quad (4.9)$$

It is easy to see that (4.9) is a multinomial coefficient generalization of (3.4). Substituting  $s \rightarrow \beta$ ,  $x \rightarrow \alpha - \beta - 1$  and  $\mathbf{z} \rightarrow \mathbf{1} + \mathbf{x}$  in (4.9), we get

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} (-1)^{|\mathbf{n}|-|\mathbf{k}|} \binom{\beta - \alpha + |\mathbf{n}|}{\mathbf{n} - \mathbf{k}} \binom{\beta + |\mathbf{k}|}{\mathbf{k}} (\mathbf{1} + \mathbf{x})^{\mathbf{k}} = \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{\alpha}{\mathbf{n} - \mathbf{k}} \binom{\beta + |\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}, \quad (4.10)$$

which is a generalization of Munarini's identity (3.6). If  $\alpha = \beta = |\mathbf{n}|$ , then (4.10) reduces to

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} (-1)^{|\mathbf{n}|-|\mathbf{k}|} \binom{|\mathbf{n}|}{\mathbf{n} - \mathbf{k}} \binom{|\mathbf{n}| + |\mathbf{k}|}{\mathbf{k}} (\mathbf{1} + \mathbf{x})^{\mathbf{k}} = \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{|\mathbf{n}|}{\mathbf{n} - \mathbf{k}} \binom{|\mathbf{n}| + |\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}},$$

which is generalization of Simons' identity (3.7). Note that Shattuck [26] and Chen and Pang [3] have given different combinatorial proofs of (3.6). It is natural to ask

*Problem 4.3* Find a combinatorial interpretation of (4.10).

## 5 Concluding remarks

We know that binomial coefficient identities usually have nice  $q$ -analogues. However, there are only curious (not natural)  $q$ -analogues of Abel's and Rothe's identities (see [25] and references therein) up to now. There seems to have no  $q$ -analogues of Jensen's identity in the literature.

It is interesting that Hou and Zeng [15] gave a  $q$ -analogue of Sun's identity (3.5):

$$\sum_{k=0}^m (-1)^{m-k} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n+k \\ a \end{bmatrix} (-xq^a; q)_{n+k-a} q^{\binom{k+1}{2} - mk + \binom{a}{2}} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m+k \\ a \end{bmatrix} x^{m+k-a} q^{mn + \binom{k}{2}}, \quad (5.1)$$

where  $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$  and

$$\begin{bmatrix} \alpha \\ k \end{bmatrix} = \begin{cases} \frac{(q^{\alpha-k+1}; q)_k}{(q; q)_k}, & \text{if } k \geq 0, \\ 0, & \text{if } k < 0. \end{cases}$$

Clearly, (5.1) may be written as a  $q$ -analogue of Munarini's identity (3.6):

$$\begin{aligned} & \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} \beta - \alpha + n \\ n - k \end{bmatrix} \begin{bmatrix} \beta + k \\ k \end{bmatrix} q^{\binom{n-k}{2} - \binom{n}{2}} (-x; q)_k \\ &= \sum_{k=0}^n \begin{bmatrix} \alpha \\ n - k \end{bmatrix} \begin{bmatrix} \beta + k \\ k \end{bmatrix} q^{\binom{n-k+1}{2} + (\beta - \alpha)(n-k)} x^k, \end{aligned} \quad (5.2)$$

as mentioned by Guo and Zeng [14]. We end this paper with the following problem.

*Problem 5.1* Is there a  $q$ -analogue of (4.10)? Or equivalently, is there a multi-sum generalization of (5.2)?

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